EXTRUDED POLYHEDRON MORPHOLOGY RESEARCH

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Abstract. An extruded polyhedron is a kind of geometry obtained by extruding the edges of a polyhedron along the normal line of its face. The interior of this geometry is a polyhedron-shaped space, and each polyhedron has a corresponding extruded polyhedron. Extruded polyhedrons formed by different polyhedrons have different properties; certain extruded polyhedrons are stable, while others are highly variable. Different variable extruded polyhedrons also have greatly different degrees of freedom. Based on previous studies, this paper thoroughly explores the deformation logic of complex extruded polyhedrons.

Keywords. Extruded polyhedron; Deformation of the logic.

1. Introduction

Researchers have always maintained great interest in the art of origami in the fields of materials, structures, nodes, etc. Structures inspired by special origami patterns have special mechanical properties, that are not present in traditional structures. Current research on origami mainly focuses on the creases in origami forms. Through special creases and crease treatments in the application process, the development of special structures can usually be realized, which is especially important in the research of deployable structures (Schenk & Guest 2013; Wei et al. 2013).

Modular origami breaks the limitation of a single paper in traditional origami art, folding multiple sheets of paper into modules; then, these modules are connected to each other to form a component. Certain forms are capable of large-scale shape transformations, making them ideal sources of inspiration for creating metamaterials, dynamic structures and components with tuned mechanical properties (Yang 2017).

The original source of extruded polyhedron geometries is a modular origami technique called Snapology. This origami technique invented by Heinz Strobl opens a new door to the study of geometry and provides a unique feature: strips are used to represent the faces and edges of polyhedrons (Strobl 2010; Goldman 2011). The extruded dodecahedron in Figure 1 is made using Snapology.

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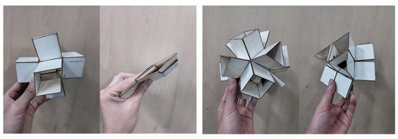
a Polyhedron b Extruded Polyhedron

a from:https://loft-concept.ru
b from:https://artekusudama.blogspot.com

Figure 1. Snapology.

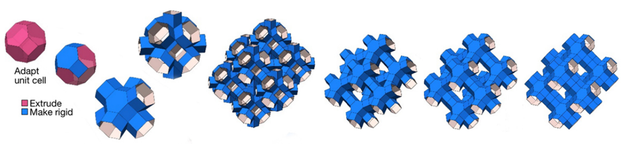
The variability of extruded polyhedrons and the application potential of modular origami methods are two important reasons for the introduction of extruded polyhedrons into scientific research. Horboman (2014) considered the extruded cube as an example for related application patents. The properties of extruded polyhedrons are completely dependent on the core polyhedral space, some of which can be deformed, while others are static. Variable extruded polyhedron coincides with the definition of an expandable structure to a certain extent (De Temmerman 2007). This is also a focus of our research.



a Deployable b Undeployable

Figure 2. Deployable structure.

Compared to the history of research on modular origami, research on extruded polyhedron, which are a special type of modular origami, remains in the exploration stage. Overvelde et al. (2017) introduced an extruded cube into the study of materials science and used it as a basic unit to perform array transformation in space. Since each unit can be deformed, the formed material can also be deformed along a predetermined path. Following this idea, the team introduced more extruded polyhedrons for research, and studied their deformation characteristics after being arrayed (Figure 3). Research has also been performed on extruded polyhedron dynamic nodes, which can be used in furniture, architecture and other applications (Yan et al. 2020).



Unit and Array
from: 《Rational design of reconfigurable prismatic architected materials》

Figure 3. Polyhedral array.

However, there are few theoretical studies of the deformation characteristics of a single expandable structure. At present, the extruded cube has been studied in detail from a mathematical point of view, and the deformation characteristics of the extruded cube are expressed by the Angle relation between the three edges located at the same vertex of the cube (Overvelde et al. 2016).

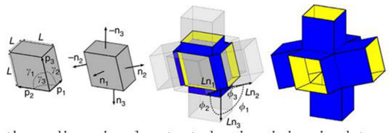


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Figure 4. Extruded hexahedron deformation.

An extruded cube presents the simplest variable extruded polyhedron, and the logic of its deformation has been described relatively completely. This deformation logic uses the concepts of vectors and angles, and can clearly express the deformation state of the extruded cube using formulae. However, these formulae also have limitations. The three edges of an extruded cube have a vertical relationship at the starting position, and its computational complexity is low. If this logic is applied to a more complex extruded polyhedron, the spatial relationship between points, edges, and faces becomes more complex, which greatly increases computational complexity. This article proposes a logically clear, cognitively simpler method to describe the deformation of complex extruded polyhedrons.

The purpose of studying these deformation characteristics is to better understand the deformation principle of extruded polyhedron and to provide theoretical support for future research of the applications of extruded polyhedron, which have potential as dynamic nodes and structures.

2. Research strategy

There are many types of polyhedrons, among which the most common are Platonic polyhedrons and Archimedes polyhedrons (Peraza-Hernandez et al. 2014). The polyhedrons we encounter in daily life can generally be classified as one of these two types. The prototype of the research object in this paper is also one of these two types of polyhedrons, and the selected polyhedrons have a certain degree of complexity and representativeness.

In our research, we set an element that can affect the deformation of an extruded polyhedron as a degree of freedom. This element is called a degree of freedom element in this article. For example, in an extruded cube, the motion of three edges at the same vertex can control the deformation of the entire cube. Then, we can say that these three edges represent three degrees of freedom. When we select three vertices, the extruded cube can deform when the positions of the three points change; thus, the three vertices can also represent three degrees of freedom. In Figure 5, points, edges, and face methods are shown as examples of degrees of freedom.

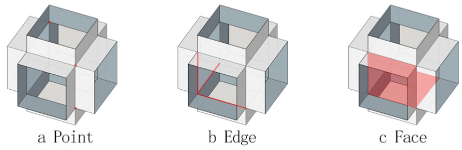


Figure 5. Degrees of freedom in different elements.

Using the angle between edges to represent the deformation characteristics of the entire extruded cube has been realized in previous research led by the Harvard University Graduate School (Overvelde et al. 2016). In this study, we discuss the deformation of extruded polyhedrons in a spherical coordinate system, where the determination of a position requires the joint action of two angles: the angle between the line and the axis between the point of interest and the origin, and the angle between the line and the plane. For convenience, the Z axis and the plane XOZ are used as references (Figure 6).

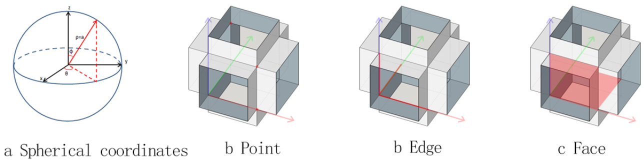


Figure 6. spherical coordinates and application.

For the three types, namely, points, edges and surfaces, the largest difference is the degree of integration of elements. One surface can integrate many points and edges, while an edge can integrate two points, and the points are scattered. The lower the degree of integration of an element, the worse the correlation between the element and other elements; the number of calculations increases accordingly. A high degree of integration of the elements will inevitably lead to a correlation between the front and back calculations; thus, the results of the previous calculations can lay the foundation for subsequent calculations, reducing the amount of calculations that must be performed.

We decided to abandon the method of using edges as degrees of freedom when investigating the logic of complex polyhedron deformation and adopted the surface method instead. Another advantage of this method is that it can divide the deformation logic into two layers. First, edges or points are used to derive the deformation of the surface, and then the deformation of the surface is used to derive the deformation of the extruded polyhedron.

3. Case study

3.1. EXTRUDED TRUNCATED-OCTAHEDRON STUDY

In this chapter, we use an extruded truncated octahedron to verify. In geometry, a truncated octahedron is a semi-regular polyhedron composed of eight hexagonal faces and six quadrilateral faces, with a total of 14 faces, 36 sides, and 24 vertices.

The deformation logic of an extruded truncated octahedron is more complex than that of an extruded cube. The face is of course the most important element of freedom. Nevertheless, the edges and points that make up the face will still run through the whole logic. Therefore, the mixed logic method dominated by surface elements is more convenient for description or calculation. The method is divided into two major steps: determine the surface, and determine the elements that are not on the surface. It is necessary to understand the role of these elements in the entire movement of the truncated octahedron. In our description system, we take the surface as the main element, and other elements are used to illustrate the shape and position of the surface.

3.1.1. .Hexagon deformation

We know that the cross section of the branches of the extruded truncated octahedron is composed of many aquadrilateral and hexagons.Obviously, the control force of the quadrilateral is far less than the control force of the hexagon to the entire extruded truncated octahedron. Then in this geometry, We use the hexagon as the degree of freedom element that controls the entire extruded truncated octahedron.

Mark the six sides as es1˜es6 , es1 is set as a fixed edge, the angle between es2 and es1 on the x0y plane is denoted as a, a in [0, π], the angle between es3 and es2 is b ,b in [π-a, π], the angle of c will cause the linkage change of the three angles d, e, f, when es5 is parallel to es6, c is at the minimum, when es5 When parallel to es5, c is at the maximum. And when c is determined, the shape of the hexagon can be directly determined.(Figure 7(a))

We will calculate the range of values of c in different values of a and b:

$$b1 = \frac{\pi - a}{2} \tag{1}$$

$$b2 = \frac{2 \cdot b + a - \pi}{2} \tag{2}$$

$$u^2 = 2 \cdot (2 - \cos a) \tag{3}$$

$$v^2 = 1 + u^2 - 2 \cdot u \cdot \cos a \tag{4}$$

$$C\min = \text{arcsin}\left(u \cdot \frac{\sin(b2)}{v}\right) + \text{arccos}\left(\frac{v^2+3}{4 \cdot v}\right) \tag{5}$$

$$C\max = \text{arcsin}\left(u \cdot \frac{\sin(b2)}{v}\right) + \text{arccos}\left(\frac{v^2-3}{2 \cdot v}\right), , a \in [0, \pi], b \in [\pi - a, \pi] \tag{6}$$

These two formulas represent the maximum and minimum values of c indifferent a and b, but the range of these two formulas will make the Angle between two edges of a hexagon bigger than 180°, so some restrictions need to be made.

When c is bigger than 180°, the minimum value formula of c is calculated according to the following formula, and the values of b1,u, and v in the formula refer to the above formula

$$f1 = \frac{\pi + a}{2} \tag{7}$$

$$b2 = \arcsin\left(\frac{\sin(f1)}{2}\right) \tag{8}$$

$$b3 = b - b1 - b2 \tag{9}$$

$$w^2 = 1 + v^2 - 2 \cdot v \cdot \cos(b3) \tag{10}$$

$$C\min 1 = \arcsin\left(v \cdot \frac{\sin(b3)}{w}\right) + \arccos\left(\frac{v^2 - 3}{2 \cdot v}\right), a \in [0, \pi], b \in [\pi - a, \pi], \tag{11}$$

When the maximum value of C is bigger than 180°, the maximum value of C is directly set to 180°

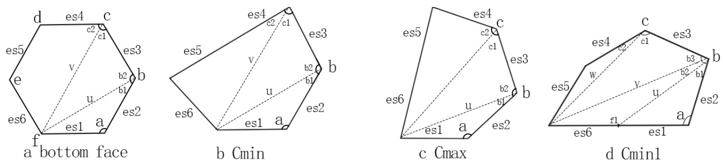


Figure 7. bottom face deformation.

By determining the position of three angles or four sides, we can determine the shape of a hexagon. In this way, we have a deep understanding of the shape of this face. The next step is to explore the influence of the parts other than the face on the entire extruded polyhedron, which is the edge or point we mentioned above.

3.1.2. other elements in deformation

After accurately describe the deformed posture of the hexagon, we use six edges to further determine the shape of the extruded truncated octahedron. However, we can also find that the interrelationship formed by these six edges determines only the other three hexagonal faces.

First, we put a hexagon on the plane XOY. For convenience, one of the sides coinciding with the X-axis and is set to be static. We thus restrict the movement of certain elements for the calculation. However, this does not affect the deformation results. For the bottom hexagon, f1 can move in the XOY plane based on the law of motion summarized in the previous section. The purple edge in Figure 8a is an element of the degree of freedom that determines the shape and position of hexagon f4 opposite to f1. Because some of the edges in the truncated octahedron are always parallel, we have optimized the choice of edges in order to facilitate the calculation because we must determine only the planes of the three faces f2, f3, and f5 to determine the position of the edge. As shown in Figure 8b, the edges are denoted e1˜e5.

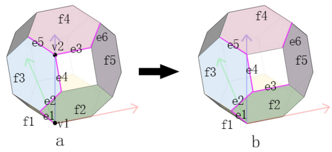


Figure 8. deformation setting.

In the coordinate system, the angle between e1 and the z axis is α, and the angle between the projection on the plane XOY and the counterclockwise phase of the XOZ plane is β. With the vector e1 = (a*sinα*cosβ, a*sinα*sinβ, a*cosα), where α in [0, 0.5 π), β in [0, π), this edge can move on the surface of the hemisphere (Figure 9). This vector is going to be used to represent the position of the f2 plane right here.

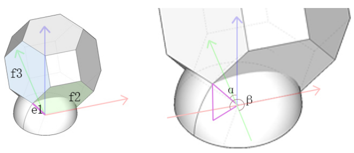


Figure 9. step 1.

When determining e1, the two edges of e1 and ex1 are sufficient to determine the plane of f2. We can calculate the range of motion of e2 and e3 on plane f2. We convert the standard plane to plane f2 when we calculate the law of motion of the edge on f2 (Figure 10). When determining e4, e5, and e6, we also need to convert the standard plane. The angle between plane f2 and plane XOZ is θ, and the included angle d1 between e1 and edge ex1 can be expressed by the known α and β:

$$\theta = \arccos\sqrt{\frac{\cos^2\alpha}{\sin^2\alpha \cdot \cos^2\beta + \cos^2\alpha}} \tag{12}$$

$$d1 = arc\cos\left(\sin\alpha \cdot \sin\beta\{r \in \mathbb{R} | 0 \leq \alpha \leq 0.5\pi\pi\}\square\{s \in \mathbb{R} | 0 \leq \beta \leq 2\pi\}\right. \tag{13}$$

Where the range of angle d2 is in [π-d1, π], and the range of angle d3 can be calculated with d1 and d2. When we obtain the exact values of these three angles, hexagon f2 can be determined.

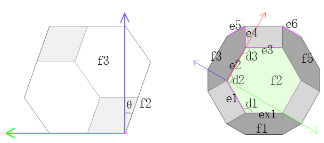


Figure 10. step 2.

When we determine the position of the edge, the shape of the entire hexagon is also determined. Therefore, this method of mixing elements can also be considered as if points and edges elements were used to determine the surface, and then the surfaces were used one by one to determine the deformed shape of the entire extruded truncated octahedron.

The determination of face f3 requires four edges. Edge ex3 is parallel to the bottom edge ex1, thus, edge ex3 has been determined. Similarly, the position of e2 has also been determined. Therefore, the angle between ex3 and e2 is already a certain angle and can be represented by r, s, and d2. The position of the face f3 is also related to the three angles r, s, and d2. Taking the edge e2 as a vector, the distance and direction from the origin to the vertex of e2 can be obtained, and the e2 vertex can be obtained, which is also the origin of the f3 plane.

In the face f3, the positions of e4 and e5 can also be determined by the limited range of angle values; e2 and ex3 have been determined in the previous edge motion; and d4 is also a given value. Thus, the range of d5 on face f3 is in [π-d4, π], and the range of d6 remains more complex and must be substituted into the range calculation formula for c shown in Section 3.1.1.

The final edge that must be determined is on face f5. At present, there are three edges that have been determined for f5 in Figure 11b, and the shape of a hexagon needs to be determined by four edges. We choose e6. The first three edges of f5 are determined in the process of determining the three faces of f1, f2, and f3. Then the range of the angle between e6 and e4 can still be calculated using the value range of c in the previous section.

Finally, through the shape of a hexagon and the position of the six edges, we determined the specific shape of the extruded truncated octahedron. From another perspective, the determination of f1 and the position of the six edges essentially determines the shape and plane of the four hexagonal faces f1, f2, f3, and f5.

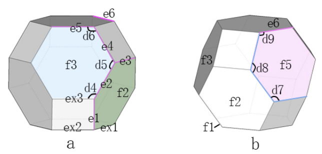


Figure 11. step 3,4.

3.2. SUMMARY

The process proposed in our research is to transform a monolithic deformation into a series of operations. By extracting elements with freedom of transformation, ignoring subordinate elements, we can determine the order of motion of the former elements, assign the constraints of the movement of the previous elements to the latter elements, and perform a stepwise analysis.

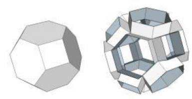
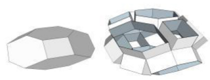
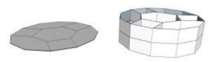
We also discovered that the final determination of our edges is in the shape of a hexagon. We can thus conclude that four hexagons can determine the shape of an extruded truncated octahedron.

To demonstrate the proposed method, we use the calculated values from our calculation process. To make it easier to understand the state of the extruded truncated octahedron, we use a table to represent a certain state. The presentation of table information is divided into two levels. The first level is the surface, and the second level comprises the three angles between the edges and the edges in the plane. The determination of each hexagon requires a spatial plane position and three angle values. Four hexagons can completely describe the shape of an extruded truncated octahedron.

In this table, each group of angle values corresponds to a shape. Except where f1 is set on the XOY plane, the locations of the planes of other polygons do not require additional explanation. The plane position of the hexagon is unique and definite. It is subject to the angle limit within the hexagon, which indicates that when we determine these angles, the plane position of the polygon and the shape of the entire geometry are definite and unique.

Table 1 Cases presentation

Table 1. The relationship between angle and form.

Plane	Anglular name			
	∠α	35.31°	65.31°	90°
	∠β	240°	120°	150°
f1	∠a	120°	120°	120°
	∠b	120°	120°	120°
	∠c	120°	120°	120°
f2	∠d1	120°	76.27°	150°
	∠d2	120°	141.87°	60°
	∠d3	120°	76.27°	150°
f3	∠d4	120°	76.27°	150°
	∠d5	120°	141.87°	60
	∠d6	120°	76.27°	150°
f5	∠d7	120°	76.27°	150°
	∠d8	120°	141.87°	60
	∠d9	120°	76.27°	150°

The table gives an example of the corresponding shape of an extruded truncated octahedron corresponding to each group of angles. The plane here can be understood as a hexagon of the plane. For the specific position, please refer to Section 3.1.2.

4. Conclusion and Prospect

This article mainly discusses the deformation logic of a complex extruded polyhedron, which we describe with mathematics in detail. However, due to the complexity of the shape, there remains a lack of precise mathematical calculations for some aspects of the shape, especially the accuracy of the plane where f3 and f5 are located. The purpose of this article is also to provide an initial understanding of a complex extruded polyhedron, which is usually considered difficult to describe, can be explained with clear logic.

It is necessary to conduct in-depth research on the deformation logic of more extruded polyhedrons, and explore the possibility of extruded polyhedrons in real applications, especially in combination with professional expertise, and how this kind of movable component can be combined with construction-related industries.

acknowledgement

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